II. PRELIMINARIES

Basic Problem

The general outline of the task is as follows:

1. The basic goal is to calculate the point of intersection between a line and plane in 3 dimensions. The line is defined by the positions of the sun and the tip of the sundial's gnomon (commonly referred to as the dial's *nodus*). The plane is the arbitrarily oriented sundial (referred to as the *dial plane*).
2. Next, the point of intersection, calculated in three dimensions, must be projected onto the 2-dimensional dial plane. This new, 2D coordinate (DP_x , DP_y), represents a single data point ultimately used in the construction of the sundial.
3. Steps 1 and 2 are repeated for each relevant solar hour angle (and all relevant solar declinations) until enough data has been calculated to construct the entire sundial.

Coodinate Systems

• 3-Dimensional
We use the familiar XYZ cartesian coordinate system (right-handed) for all calculations (see Fig. 1). The horizontal plane at the location of the sundial represents the X-Y plane: the positive x-axis is due East, while the positive y-axis is due North. The vertical represents the positive z-axis. All references to "3D" coordinates will use this representation. The azimuth and altitude of the sun are measured as shown in Figure 3.

• 2-Dimensional
Any reference to 2D will refer to the *plane of the dial* (see Fig. 2). It represents the surface on which the hour lines are ultimately to be drawn. We again use a right-handed cartesian coordinate system. The x-axis is always horizontal, even for arbitrarily oriented dial planes. For a horizontal dial plane (the one with which most people are familiar), the 2D x-axis and y-axis coincide with the 3D x and y axes, respectively. For an arbitrarily oriented dial plane, the x-axis is defined as the intersection between the dial plane and a horizontal plane passing through the origin. *The x-axis of the dial plane will, in general, be a horizontal line in the 3D system, confined to the x-y plane and passing through the origin*. The y-axis is always perpendicular to the x-axis.

• Origin
The origin, whether 3D or 2D, is defined as the point at which the gnomon (or style) attaches to the dial face. The origin of the 2D system and the origin of the 3D system *represent the same physical location*.

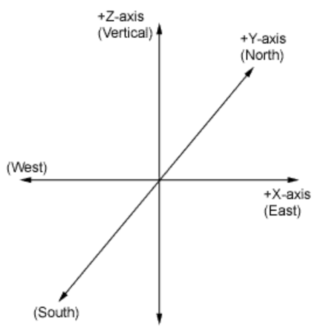

Fig. 1: XYZ Axes

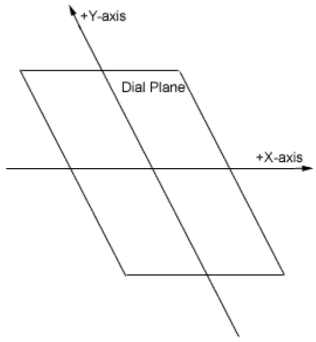

Fig. 2: 2D Dial Plane

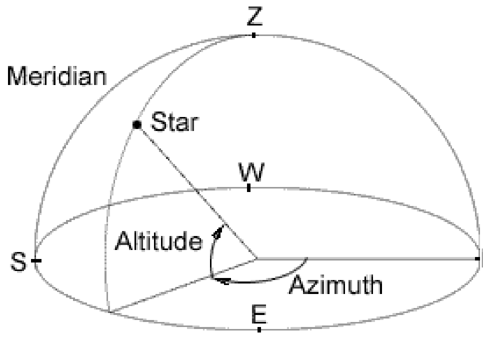

Fig. 3: Azimuth and Altitude

Dial Face Inclination & Declination

When most people think of a sundial, they envision an object with a horizontal plane and a gnomon of some type, usually slanted, attached at a right angle. But there is nothing to prevent a dial maker from orienting the dial face in a completely arbitrary direction. In general, a dial may be tilted with respect to the vertical (inclination) and rotated azimuthally (declination). This last term should not to be confused with the declination associated with *coordinates on the Celestial Sphere*. Dial face inclination is defined as the angle measured from the vertical to the dial face normal (see Fig. 4). Dial face declination is the angle measured in the horizontal plane, from due south to the vertical plane which contains the dial's zenith and surface normal (see Fig. 5). Declinations east of south are negative; west of south, positive. The following table should help to illustrate how each measurement is defined.

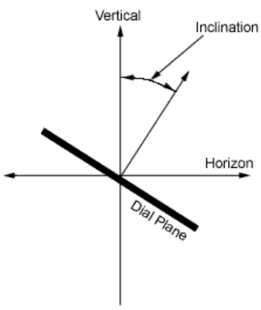

Fig. 4: Dial Plane Inclination

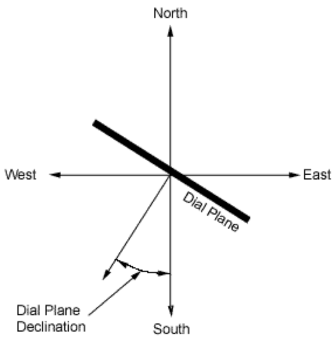

Fig. 5: Dial Plane Declination

Dial Orientation	Inclination	Declination
horizontal	0°	0°
vertical, south-facing	90°	0°
vertical, east-facing	90°	-90°
vertical, west-facing	90°	90°
vertical, facing south-west	90°	45°

Spherical Triangle (PZS)

This article does not discuss spherical trigonometry, nor how to solve the PZS triangle. There is already a lot of information available on the web for this, so there is no point in re-inventing the wheel. It is assumed you can already determine the sun's altitude and azimuth at your latitude for the relevant solar hour angles needed to produce a working sundial.

The altitude and azimuth of the sun are used to calculate the direction cosines of the sun's position, which are then used in determining the equation of the line connecting the sun and the dial's nodus.

III. CALCULATIONS

In general, calculations may be classified as falling into one of two categories: 1) those which are required only once; and 2) those which must be performed for each position of the sun for which a dial layout data point is desired. In addition, throughout this article no attempt is made to "simplify" the various unit vectors or formulae to expressions involving the basic initial inputs: latitude, dial inclination and declination, and solar azimuth and altitude. In many instances, the process of "simplification" leads to longer, more cumbersome equations, although this is not always the case. Purists are free to disagree and rewrite equations as they see fit.

One-Time Calculations

1 :: Equation and Unit Normal of the Dial Plane

The general equation describing a plane in three dimensions is given by Ax + By + Cz = D. By defining the dial plane as always passing through the origin, this simplifies to:

$$Ax + By + Cz = 0$$

... Eq. 1

In addition, a very useful property of the general equation is that the surface normal of the plane is given by:

$$\hat{\mathbf{n}} = \frac{A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}}{\sqrt{A^2 + B^2 + C^2}}$$

... Eq. 2a

If the coefficients A, B and C are the direction cosines of the vector **n**, the equation reduces to:

$$\hat{\mathbf{n}} = A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}$$

... Eq. 2b

If we know the inclination and declination of the dial plane, we can easily calculate the coefficients as follows:

x-direction : A = -sin(I) sin(Δ)
y-direction : B = -sin(I) cos(Δ)
z-direction : C = cos(I)
where I = the inclination of the dial plane
and Δ = the declination of the dial plane

2 :: Dial Plane X-axis.

Earlier it was discussed that the x-axis of the dial plane represents a line which passes through the 3D origin and is confined to the horizontal, or 3D x-y, plane. To determine the equation of this line (in the 3D system), we want to find the *intersection between the horizontal plane (i.e. z=0) and the plane of the dial (Ax + By + Cz = 0)*. The line of intersection between two planes can be determined by taking the **cross product** of the unit normals of each plane. The unit normal of the dial plane was calculated earlier and the unit normal of the 3D, x-y plane is **k**, so the vector **h**, representing the dial plane x-axis, can be determine as follows:

$$\vec{\mathbf{h}} = \hat{\mathbf{k}} \times \hat{\mathbf{n}} = \hat{\mathbf{k}} \times \left(A\hat{\mathbf{i}} + B\hat{\mathbf{j}} + C\hat{\mathbf{k}}\right)$$

... Eq. 3a

Here, **h** represents a vector in the x-y plane of the 3D coordinate system that also lies completely in the 2D plane of the dial. Normalizing the vector gives the following:

$$\hat{\mathbf{h}} = \frac{-B\hat{\mathbf{i}} + A\hat{\mathbf{j}}}{\sqrt{A^2 + B^2}}$$

... Eq. 3b

3 :: Position Vector and Coordinates of the Dial's Nodus.

For a classic sundial, one may, in general, choose between two different gnomon configurations: 1) the gnomon may be a perpendicular rod attached at right angles to the dial plane; or 2) the gnomon may be attached to the dial plane such that the style (or axis) of the gnomon is parallel to the Earth's rotational axis (i.e. a *polar* gnomon). Each choice requires a slightly different approach in calculating the necessary parameters.

CASE I : PERPENDICULAR GNOMON

This case is relatively trivial since the unit normal of the dial face also defines the direction of a perpendicular gnomon. The position vector of the nodus is just the length of the gnomon (G) multiplied by the unit normal (Eq. 2b) to the dial plane:

$$\vec{\mathbf{g}} = G\hat{\mathbf{n}} = GA\hat{\mathbf{i}} + GB\hat{\mathbf{j}} + GC\hat{\mathbf{k}}$$

... Eq. 4a

Case II : Polar Gnomon

This case is a little more complicated than Case I, but is still straight-forward. We make use of the fact that a polar gnomon is restricted to the y-z plane (i.e. x=0). This yields the following two possibilities:

$$\vec{g} = G\left(0\hat{i} + \cos(lat)\hat{j} + \sin(lat)\hat{k}\right) \qquad \text{... Eq. 4b}$$

or,

$$\vec{g} = G\left(0\hat{i} - \cos(lat)\hat{j} - \sin(lat)\hat{k}\right) \qquad \text{... Eq. 4c}$$

For a horizontal dial, clearly the first equation applies; for a vertical, south-facing dial, the second. But for an arbitrarily oriented dial plane, the choice can be made by using the dot product to *determine which vector representation of **g** lies in front of the plane of the dial*, as follows:

> Whichever representation of **g** yields **g** • **n** > 0 is the correct choice. If the dot product is positive, the angle between the dial plane normal and the nodus' position vector is < 90°, which it must be in order to cast a shadow. If the dot product is negative, this means the angle between the two vectors is > 90°, which is an unworkable configuration and means the other representation for **g** should be used.

4 :: Determination of the Substyle

Installation of perpendicular gnomons, or polar gnomons when the dial plane is horizontal (or vertical & south-facing), hardly need mentioning except to reiterate the simplicity of it. The situation is not so simple when the dial plane is oriented at some unusual combination of inclination and declination. Figure 6 shows the general orientation of a dial plane (defined by unit normal **n**) with the nodus' position vector, **g**. The dial plane x-axis is also defined by unit vector **h** (previously defined, see Eq. 3b). In order to install a gnomon, one needs to know the following:

1. *style height*: this is the shortest (or perpendicular) distance from the nodus to the dial plane. In Figure 6, this distance is equal to the length of vector **t**.
2. *substyle coordinates*: these coordinates represent the point of intersection, in the plane of the dial, between vector **t** and the dial plane. In Figure 6, these coordinates are represented by $\mathbf{sb_x}$ and $\mathbf{sb_y}$.

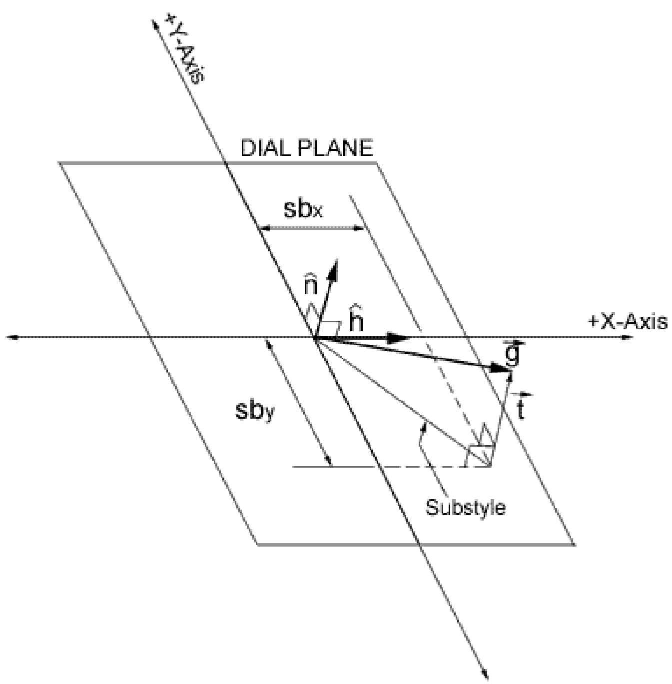

Fig. 6: Style and Substyle

From the figure, it should be clear that **t** represents the *component of **g** parallel to the dial plane unit normal*. The magnitude of **t** is simply **g** dotted onto the dial plane unit normal, **n**. So the substyle height becomes |**t**| = **g** • **n**. Equation 5a gives the full expansion.

$$\text{style height} = \left|\vec{t}\right| = \vec{g} \cdot \hat{n} = Ag_x + Bg_y + Cg_z \qquad \text{... Eq. 5a}$$

In order to calculate the coordinates of the substyle, we need to determine the magnitude of the components of vector **g** along the x and y axes of the dial plane. This can be accomplished by dotting **g** onto unit vectors defining each axis. We have already defined a unit vector, **h**, in the direction of the dial plane x-axis, so sb_x is simply **g** • **h** (see Eq. 5b).

The cross product of the two unit vectors **n** and **h** results in a third unit vector orthogonal to both. This third vector represents the y-axis of the dial plane. When we dot **g** onto the unit vector produced by the cross product **n** × **h**, we obtain the triple scalar product **g** • **n** × **h**, the value of which is the substyle coordinate, sb_y (see Eq. 5c).

$$sb_x = \vec{g} \cdot \hat{h} = \left(g_x\hat{i} + g_y\hat{j} + g_z\hat{k}\right) \cdot \frac{-B\hat{i} + A\hat{j}}{\sqrt{A^2 + B^2}} = \frac{Ag_y - Bg_x}{\sqrt{A^2 + B^2}} \qquad \text{... Eq. 5b}$$

$$sb_y = \vec{g}\cdot\hat{n}\times\hat{h} = \frac{\det\begin{pmatrix} g_x & g_y & g_z \\ A & B & C \\ -B & A & 0 \end{pmatrix}}{\sqrt{A^2+B^2}} = \frac{(A^2+B^2)g_z - (Ag_x + Bg_y)C}{\sqrt{A^2+B^2}}$$ ... Eq. 5c

NOTE: For horizontal dial planes, the coefficients A and B are both zero, resulting in undefined values for Equations 5b and 5c. Horizontal dials are relatively easy to calculate and construct, so no space will be devoted here to considering the required modifications to Eqs. 5b and 5c when dealing with this type.

5 :: Determination of the Noon-Line

The location of the noon-line (i.e. when the sun transits the local meridian) can be thought of as the intersection between the dial plane and the local merdian with the added constraint that the plane of the meridian must also pass through the dial's nodus. Within the plane of the dial, the noon-line will have the familiar form *y = mx + b*, while noting that for polar gnomons, the y-intercept, *b*, will always be zero (i.e. the noon-line will always pass through the origin). The direction of the line of intersection, in three dimensions, between the dial plane (unit normal given by Eq. 2b) and meridian (unit normal of **i**) can be determined by taking the cross-product of their unit normals as shown in Eq. 6a:

... Eq. 6a

$$\vec{m} = \hat{i}\times\hat{n} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ A & B & C \end{pmatrix} = -C\hat{j} + B\hat{k}$$

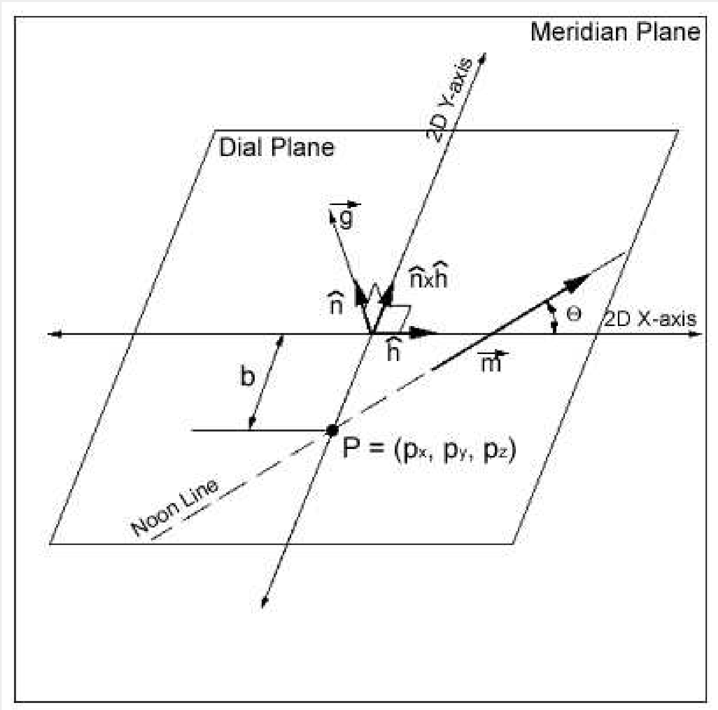

Fig. 7: Intersection of Dial Plane and Local Meridian

Since Eq. 6a represents the noon-line direction in 3D, we must now convert it to its representation in the 2D dial plane. We can determine the slope, tan(θ), by first calculating the component of **m** along each dial plane axis (i.e. **h** and **n × h**) and then taking the ratio. This operation as is shown in Eq. 6b:

... Eq. 6b

$$\tan(\theta) = \frac{\vec{m}\cdot\left(\hat{n}\times\hat{h}\right)}{\vec{m}\cdot\hat{h}} = \frac{\det\begin{pmatrix} 0 & -C & B \\ A & B & C \\ -B & A & 0 \end{pmatrix}}{\left(-C\hat{j} + B\hat{k}\right)\cdot\left(-B\hat{i} + A\hat{j}\right)} = -\frac{B}{AC}$$

For polar gnomons, Eq 6b is sufficient to plot the noon-line; however, in the case of perpendicular gnomons, the y-intercept must also be determined. We know the 3D equation of the meridian is little more than $x = g_x$ (i.e. every point on the meridian has the same x-coordinate, namely g_x, since the nodus must lie in this plane). First, let's denote the 3D point on the dial plane y-axis where it intersects the merdian as P. Now, the representation of a line in three dimensions can take the form shown in Eq. 6c, where (x_o , y_o , z_o) represents a point through which the line passes and (ε_x , ε_y , ε_z) represent the direction cosines of the line. Next, since P is on the dial plane y-axis, the point (x_o , y_o , z_o) can represent the origin, (0, 0, 0), and (ε_x , ε_y , ε_z) represents the direction cosines of the y-axis. We also know the 3D x-coordinate of P is g_x because it must lie in the plane of the meridian. Taking advantage of all this, the entire expression in Eq. 6c reduces to Eq. 6d, from which we can immediately solve for P_y and P_z, which is shown in Eq. 6e.

$$\frac{x - x_o}{\epsilon_x} = \frac{y - y_o}{\epsilon_y} = \frac{z - z_o}{\epsilon_z}$$ ... Eq. 6c

$$\frac{g_x}{\epsilon_x} = \frac{P_y}{\epsilon_y} = \frac{P_z}{\epsilon_z} \quad \text{... Eq. 6d}$$

$$P_y = \frac{\epsilon_y}{\epsilon_x} g_x \cdot \cdot \cdot \text{and} \cdot \cdot \cdot P_z = \frac{\epsilon_z}{\epsilon_x} g_x \quad \text{... Eq. 6e}$$

At this point, it can be seen that the distance from the origin to point P is nothing more than the y-intercept, *b*, and can be calculated directly from point P's components as shown in Eq. 6f. The seemingly arbitrary introduction of the minus sign in the final equation stems from the result that calculating b from the sum of the squares of P's components destroys any information about the appropriate sign (i.e. whether ± applies). An alternative calculation, which is more labor intensive, but *does* preserve the sign, involves dotting the vector pointing *from* the nodus *to* point P onto the dial plane y-axis. Mathematically, the expression would look something like this: (**g** - **P**) • (**n** × **h**). Working through the algebra results in the same expression, but with a minus sign.

$$b = \sqrt{P_x^2 + P_y^2 + P_z^2} = \frac{g_x}{\epsilon_x}\sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2} = \frac{g_x}{\epsilon_x} = G\frac{\sqrt{A^2 + B^2}}{C} \quad \text{... Eq. 6f}$$

$$b = -G\frac{\sqrt{A^2 + B^2}}{C}$$

In closing this section, it is probably worthwhile to point out that taking time to simplify Eqs. 6b & 6f results in $\tan(\theta) = [\tan(I)*\cos(\Delta)]^{-1}$ and b = -G*tan(I), where I = *dial plane inclination*, and Δ = *dial plane declination*. These expressions may look familiar to some readers.

Repetitive Calculations

This section outlines the calculations required for each solar position (i.e. each combination of *hour angle* and *solar declination*) used in laying out the sundial.

1 :: The Sun's Position Vector.
This operation is straight-forward and is merely a determination of the sun's direction cosines from its (previously determined) altitude and azimuth:

Direction cosines:
x-dir : α = cos(*alt*) sin(*azm*)
y-dir : β = cos(*alt*) cos(*azm*)
z-dir : γ = sin(*alt*)
where, *alt* = solar altitude
and *azm* = solar azimuth

The unit position vector becomes:

$$\hat{\mathbf{s}} = \alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}} + \gamma\hat{\mathbf{k}} = \cos(\text{alt})\sin(\text{azm})\hat{\mathbf{i}} + \cos(\text{alt})\cos(\text{azm})\hat{\mathbf{j}} + \sin(\text{alt})\hat{\mathbf{k}} \quad \text{... Eq. 7}$$

It should be noted that unit vector **s** points *from* the dial *toward* the sun. This representation is an important consideration in the next section.

2 :: Checking Visibility of the Sun
Given the dial plane may have any orientation (i.e. inclined and/or declined), one cannot be sure the sun will shine on the dial plane just because it is above the horizon (e.g. a vertical, west-facing dial plane does not "see" the sun until it has transited the local meridian). For a particular solar position to cast a shadow on the dial plane, the sun must be both: 1) *above the horizon*; and 2) *in front of the dial plane*. We can quickly and easily check for solar visibility by calculating a single dot product for each condition:

- If the dot product between the solar position vector and the vertical is positive (Eq. 8a), the first condition is satified, and the sun is above the horizon.
- Similarly, if the dot product between the solar position vector and the dial plane normal is positive (Eq. 8b), the second condition is satisfied, and the sun is *in front of* the dial plane.

$$\hat{\mathbf{s}} \cdot \hat{\mathbf{k}} > 0 \quad \text{... Eq. 8a}$$

$$\hat{\mathbf{s}} \cdot \hat{\mathbf{n}} > 0 \qquad \text{... Eq. 8b}$$

When using the outline of calculations presented in this article as part of an executable program, these two very simple checks will quickly discriminate between solar positions which do (or don't) cast a shadow on the dial plane. Even though the amount of time saved is probably small when using today's average computer, it is still a good idea to use them. For spreadsheet applications, however, this step may not be applicable.

3 :: The Nodus-Solar Line.
While the direction cosines of the solar position are known, in order to completely specify the 3D line which casts the shadow on the dial, one must also specify a unique point through which it passes. In this case, that unique point is the dial's nodus. Generically, we define the position of the nodus by the coordinates (g_x , g_y , g_z), which are calculated in a manner consistent with the earlier outline. The sun also has the calculated direction cosines (α, β, γ). Using this information, the line connecting the nodus and sun is defined by the following set of equations:

$$\frac{x - g_x}{\alpha} = \frac{y - g_y}{\beta} = \frac{z - g_z}{\gamma} \qquad \text{... Eq. 9}$$

4 :: Point of Intersection between the Nodus-Solar Line and the Dial Plane.
We are now ready to calculate the 3D-coordinates of the position where the nodus' shadow strikes the dial plane for the given solar position. Solving equations 1 and 9 simultaneously yield the following (where we are using the letter "c" to denote the coordinates of the actual point of intersection):

$$c_z = \frac{(\alpha A + \beta B)g_z - \gamma(Ag_x + Bg_y)}{\alpha A + \beta B + \gamma C} \qquad \text{... Eq. 10a}$$

$$c_x = \frac{\alpha}{\gamma}(c_z - g_z) + g_x \qquad \text{... Eq. 10b}$$

$$c_y = \frac{\beta}{\gamma}(c_z - g_z) + g_y \qquad \text{... Eq. 10c}$$

All of the quantities on the right-hand side of Eq. 10a are known, so it is now possible to calculate the z-axis coordinate, c_z, of the point of intersection. Substituting back into equations 10b and 10c yields c_x and c_y, respectively. This coordinate, (c_x , c_y , c_z), can now be used to determine the desired coordinates, (DP_x , DP_y), in the plane of the dial.

5 :: Dial Plane Coordinate → DP_x.

Before proceeding, it is worthwhile to be reminded that the position vector of the point of intersection calculated in the previous step, **c** = c_x**i** + c_y**j** + c_z**k**, is a vector which *lies in the plane of the dial*. Realizing this, it becomes a relatively simple matter to determine DP_x by taking the dot product of the 3D coordinate position vector **c** with the unit vector, **h**, which represents the x-axis in the plane of the dial (See Fig. 8). The operation yields the component of **c** along the dial plane's x-axis, which is the dial plane x-coordinate DP_x (see Eq. 11).

... Eq. 11

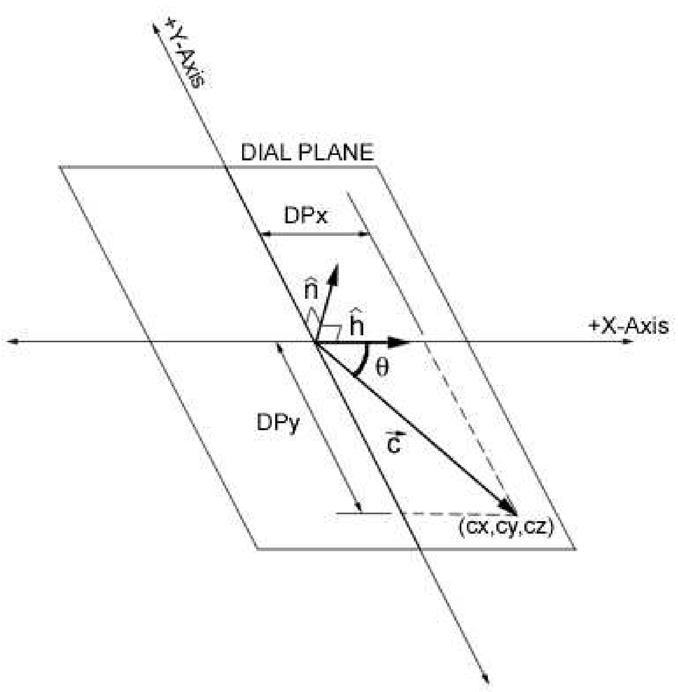

Fig. 8: Dial Plane Coordinates (DP_x , DP_y)

$$DP_x = \vec{c}\cdot\hat{h} = \left(c_x\hat{i} + c_y\hat{j} + c_z\hat{k}\right)\cdot\frac{-B\hat{i} + A\hat{j}}{\sqrt{A^2+B^2}} = \frac{-Bc_x + Ac_y}{\sqrt{A^2+B^2}}$$

6 :: Dial Plane Coordinate → DP_y.

Computation of the 2D y-coordinate, DP_y, is only slightly more complicated than the previous step. We have previously noted that cross product **n × h** results in a unit vector parallel to the dial plane's y-axis. The desired coordinate, DP_y, is merely the result of dotting **c** onto this unit vector (**n × h**). The resulting triple scalar product is shown in Eq. 12, along with its evaluation in terms of known quantities.

$$DP_y = \vec{c}\cdot\hat{n}\times\hat{h} = \frac{\det\begin{pmatrix} c_x & c_y & c_z \\ A & B & C \\ -B & A & 0\end{pmatrix}}{\sqrt{A^2+B^2}} = \frac{(A^2+B^2)c_z - (Ac_x - Bc_y)C}{\sqrt{A^2+B^2}} \quad \text{... Eq. 12}$$

IV. CONCLUSION

For somewhat difficult to calculate sundials (i.e. inclined/declined) with either polar or perpendicular gnomons, we find that a technique using primarily vector operations can yield a surprisingly simple set of calculations. These are summarized in the table shown below.

One final remark is worth making: the approach taken in the article, the use of vector notation and operators, can easily be extended to bifilar sundials. Readers interested in pursuing the use of vectors to calculate this type of dial are encouraged to read Bifilar Sundial Vector Theory and to use (and download) the companion bifilar sundial calculator.

Summary of Calculations & Results

TOP An interesting result can be had for **perpendicular gnomon dials** when one performs all the necessary algebraic substitution to get an equation in terms of the direction cosines for the dial plane unit normal (A, B, C), the solar position unit vector (α, β, γ) and the length of the gnomon, G. These equations are nearly identical to the simplified expressions for bifilar sundials. Readers who take the trouble to look at the bifilar dial article may note that when the thread offsets, O_H & O_V, are the same height, the equations are *identical* for the simple reason that similar thread heights (i.e. $O_H = O_V$) are the functional equivalent of a perpendicular gnomon dial.	**Perpendicular Gnomon *ONLY*** $DP_x = \frac{\alpha B - \beta A}{\psi\sqrt{A^2+B^2}}\,G$ $DP_y = \frac{\psi C - \gamma}{\psi\sqrt{A^2+B^2}}\,G$ where, $\psi = \alpha A + \beta B + \gamma C$
A similar approach can be taken with **polar gnomon dials**. Performing all the necessary algebraic substitution to get an equation in terms of the direction cosines for the dial plane unit normal (A, B, C), the solar position unit vector (α, β, γ) and the coordinates of the tip of the polar gnomon (0, ±Gcos(lat), ±Gsin(lat)) yields the simplified formulas shown to the right. Note that σ is merely the dot product between the dial plane unit normal, **n**, and the nodus' position vector, **g**. Also, these formulas are only valid for *inclined* dials. The $\sqrt{(A^2 + B^2)}$ term in the denominator "blows up" (i.e. it approaches infinity as the inclination goes to 0° and the dial becomes horizontal). For horizontal dials, both A and B equal 0.	**Polar Gnomon *ONLY*** $DP_x = \frac{\psi A g_y + \sigma(\alpha B - \beta A)}{\psi\sqrt{A^2+B^2}}$ $DP_y = \frac{\psi g_z - \gamma\sigma}{\psi\sqrt{A^2+B^2}}$ where, $\psi = \alpha A + \beta B + \gamma C$, and $\sigma = \hat{n}\cdot\vec{g} = Ag_x + Bg_y + Cg_z$

Step	Description	Calculation
1†	Determination of dial plane coefficients using the values of *inclination* and *declination*.	A = -sin(I)sin(Δ) B = -sin(I)cos(Δ) C = cos(I) where I = the inclination of the dial plane and Δ = the declination of the dial plane
2†	Determine the coordinates of the sundial's nodus. (actual values are dependent on initial dial parameters)	Generalized Coordinates: (g_x , g_y , g_z) Perpendicular Gnomon: (GA , GB , GC)

		Polar Gnomon: (0, Gcos(*lat*) , Gsin(*lat*)) or (0, -Gcos(*lat*) , -Gsin(*lat*)) where G = gnomon (or style) length
3†	Determine the style height.	$\text{style height} = Ag_x + Bg_y + Cg_z$
4†	Determine the substyle coordinates (the substyle distance and angle are easily calculated from these quantities) NOTE: these calculations will only be needed for polar gnomon dials.	$sb_x = \dfrac{Ag_y - Bg_x}{\sqrt{A^2 + B^2}}$ $sb_y = \dfrac{(A^2 + B^2)g_z - (Ag_x + Bg_y)C}{\sqrt{A^2 + B^2}}$
5†	Determine the slope, tan(θ), and y-intercept, *b* of the dial plane noon-line.	$\tan(\theta) = -\dfrac{B}{AC}$ $b = -G\dfrac{\sqrt{A^2 + B^2}}{C}$
6‡	Determine the solar direction cosines.	α = cos(*alt*) × sin(*azm*) β = cos(*alt*) × cos(*azm*) γ = sin(*alt*) where, *alt* = solar altitude and *azm* = solar azimuth
7‡	Determine the 3D coordinates (c_x , c_y , c_z) of the point of intersection between the solar-nodus line and the dial plane.	$c_z = \dfrac{(\alpha A + \beta B)g_z - \gamma(Ag_x + Bg_y)}{\alpha A + \beta B + \gamma C}$ $c_y = \dfrac{\beta}{\gamma}(c_z - g_z) + g_y$ $c_x = \dfrac{\alpha}{\gamma}(c_z - g_z) + g_x$
8‡	Determine the dial plane coordinates (DP_x , DP_y).	$DP_x = \dfrac{-Bc_x + Ac_y}{\sqrt{A^2 + B^2}}$ $DP_y = \dfrac{(A^2 + B^2)c_z - (Ac_x - Bc_y)C}{\sqrt{A^2 + B^2}}$

† : "one time" calculation.

‡ : calculation required for each plottable solar position.